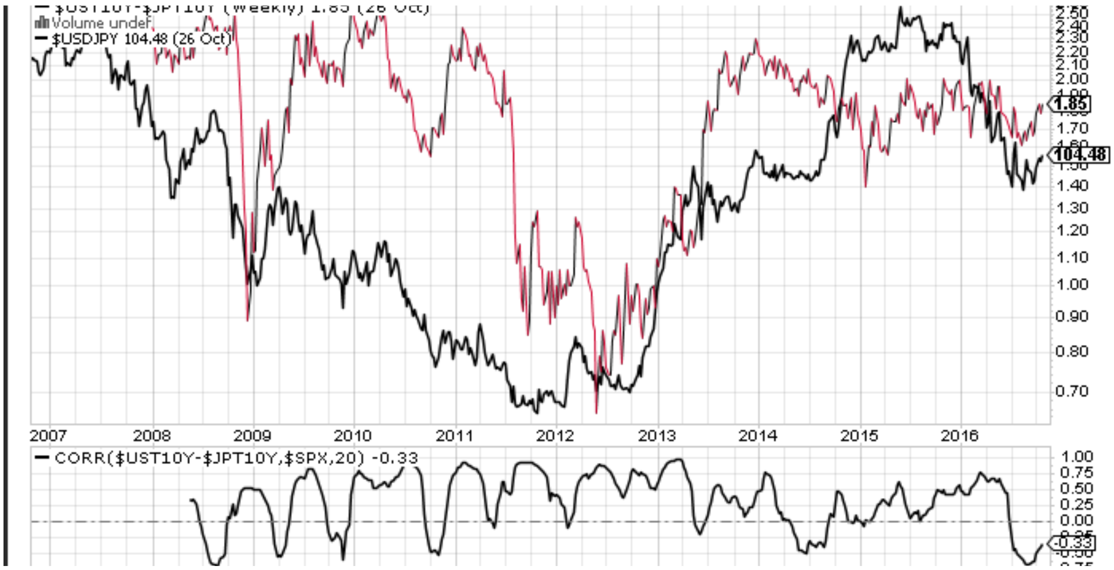
☰

⌕



ALTEXPLOIT 2018-10-10

DERIVATIVE PRICING THEORY: CALL, PUT OPTIONS AND "BLACK, SCHOLES'" HEDGED PORTFOLIO

ECONOFICTION BLACK SCHOLES, DERIVATE, FINANCE, PRICING THEORY

$$
\begin{aligned}
V(S,t) = {} & \frac{1}{2} S \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(K) + 2 \ln(S) + \sigma^2 T - \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) \\
& - \frac{1}{2} \mathbf{e}^{(-(T-t) r)} K \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(K) + 2 \ln(S) - \sigma^2 T + \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) \\
& - \frac{1}{2} S \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(2) - 2 \ln(K) + 2 \ln(S) + \sigma^2 T - \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) \\
& + \frac{1}{2} \mathbf{e}^{(-(T-t) r)} K \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(2) - 2 \ln(K) + 2 \ln(S) - \sigma^2 T + \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) \\
& + \frac{1}{2} S a \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(2) - 2 \ln(K) + 2 \ln(S) + \sigma^2 T - \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) \\
& - \frac{1}{2} \mathbf{e}^{(-(T-t) r)} a K \operatorname{erf}\left(\frac{1}{4} \frac{\sqrt{2}\,(2 r T - 2 r t - 2 \ln(2) - 2 \ln(K) + 2 \ln(S) - \sigma^2 T + \sigma^2 t)}{\sqrt{T-t}\,\sigma}\right) + \frac{S a}{2} - \frac{1}{2} \mathbf{e}^{(-(T-t) r)} a K
\end{aligned}
$$

Fischer Black and Myron Scholes revolutionized the pricing theory of options by showing how to hedge continuously the exposure on the short position of an option. Consider the writer of a call option on a risky asset. S/he is exposed to the risk of unlimited liability if the asset price rises above the strike price. To protect the writer's short position in the call option, s/he should consider purchasing a certain amount of the underlying asset so that the loss in the short position in the call option is offset by the long position in the asset. In this way, the writer is adopting the hedging procedure. A hedged position combines an option with its underlying asset so as to achieve the goal that either the asset compensates the option against loss or otherwise. By adjusting the proportion of the underlying asset and option continuously in a portfolio, Black and Scholes demonstrated that investors can create a riskless hedging portfolio where the risk exposure associated with the stochastic asset price is eliminated. In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn an expected rate of return equal to the

riskless interest rate.

Black and Scholes made the following assumptions on the financial market.

1. Trading takes place continuously in time.
2. The riskless interest rate r is known and constant over time.
3. The asset pays no dividend.
4. There are no transaction costs in buying or selling the asset or the option, and no taxes.
5. The assets are perfectly divisible.
6. There are no penalties to short selling and the full use of proceeds is permitted.
7. There are no riskless arbitrage opportunities.

The stochastic process of the asset price S_t is assumed to follow the geometric Brownian motion

$dS_t/S_t = \mu\ dt + \sigma\ dZ_t$ –– (1)

where μ is the expected rate of return, σ is the volatility and Z_t is the standard Brownian process. Both μ and σ are assumed to be constant. Consider a portfolio that involves short selling of one unit of a call option and long holding of Δ_t units of the underlying asset. The portfolio value $\Pi (S_t, t)$ at time t is given by

$\Pi = -c + \Delta_t S_t$ –– (2)

where $c = c(S_t, t)$ denotes the call price. Note that Δ_t changes with time t, reflecting the dynamic nature of hedging. Since c is a stochastic function of S_t, we apply the ***Ito lemma*** to compute its differential as follows:

$dc = \partial c/\partial t\ dt + \partial c/\partial S_t\ dS_t + \sigma^2/2\ S_t^2\ \partial^2 c/\partial S_t^2\ dt$

such that

$-dc + \Delta_t\ dS_t = (-\partial c/\partial t - \sigma^2/2\ S_t^2\ \partial^2 c/\partial S_t^2)dt + (\Delta_t - \partial c/\partial S_t)dS_t$

$= [-\partial c/\partial t - \sigma^2/2\ S_t^2\ \partial^2 c/\partial S_t^2 + (\Delta_t - \partial c/\partial S_t)\mu S_t]dt + (\Delta_t - \partial c/\partial S_t)\sigma S_t\ dZ_t$

The cumulative financial gain on the portfolio at time t is given by

$G(\Pi (S_t, t)) = \int_0^t -dc + \int_0^t \Delta_u\ dS_u$

$= \int_0^t [-\partial c/\partial u - \sigma^2/2\ S_u^2\ \partial^2 c/\partial S_u^2 + (\Delta_u - \partial c/\partial S_u)\mu S_u]du + \int_0^t (\Delta_u - \partial c/\partial S_u)\sigma S_u\ dZ_u$ –– (3)

The stochastic component of the portfolio gain stems from the last term, $\int_0^t (\Delta_u - \partial c/\partial S_u)\sigma S_u\ dZ_u$. Suppose we adopt the dynamic hedging strategy by choosing $\Delta_u = \partial c/\partial S_u$ at all times u < t, then the financial gain becomes deterministic at all times. By virtue of no arbitrage, the financial gain should be the same as the gain from investing on the risk free asset with dynamic position whose value equals $-c + S_u \partial c/\partial S_u$. The deterministic gain from this dynamic position of riskless asset is given by

$M_t = \int_0^t r(-c + S_u \partial c/\partial S_u)du$ –– (4)

By equating these two deterministic gains, $G(\Pi (S_t, t))$ and M_t, we have

$-\partial c/\partial u - \sigma^2/2\ S_u^2\ \partial^2 c/\partial S_u^2 = r(-c + S_u \partial c/\partial S_u),\ 0 < u < t$

which is satisfied for any asset price S if c(S, t) satisfies the equation

$\partial c/\partial t + \sigma^2/2\ S^2\ \partial^2 c/\partial S^2 + rS\partial c/\partial S - rc = 0$ –– (5)

This parabolic partial differential equation is called the Black–Scholes equation. Strangely, the parameter μ, which is the expected rate of return of the asset, does not appear in the equation.

To complete the formulation of the option pricing model, let's prescribe the auxiliary condition. The terminal payoff at time T of the call with strike price X is translated into the following terminal condition:

$c(S, T) = \max(S - X, 0)$ –– (6)

for the differential equation.

Since both the equation and the auxiliary condition do not contain ρ, one concludes that the call price does not depend on the

actual expected rate of return of the asset price. The option pricing model involves five parameters: S, T, X, r and σ. Except for the volatility σ, all others are directly observable parameters. The independence of the pricing model on μ is related to the concept of risk neutrality. In a risk neutral world, investors do not demand extra returns above the riskless interest rate for bearing risks. This is in contrast to usual risk averse investors who would demand extra returns above r for risks borne in their investment portfolios. Apparently, the option is priced as if the rates of return on the underlying asset and the option are both equal to the riskless interest rate. This risk neutral valuation approach is viable if the risks from holding the underlying asset and option are hedgeable.

The governing equation for a put option can be derived similarly and the same Black–Scholes equation is obtained. Let V (S, t) denote the price of a derivative security with dependence on S and t, it can be shown that V is governed by

$\partial V/\partial t + \sigma^2/2\, S^2\, \partial^2 V/\partial S^2 + rS\partial V/\partial S - rV = 0$ —– (7)

The price of a particular derivative security is obtained by solving the Black–Scholes equation subject to an appropriate set of auxiliary conditions that model the corresponding contractual specifications in the derivative security.

The original derivation of the governing partial differential equation by Black and Scholes focuses on the financial notion of riskless hedging but misses the precise analysis of the dynamic change in the value of the hedged portfolio. The inconsistencies in their derivation stem from the assumption of keeping the number of units of the underlying asset in the hedged portfolio to be instantaneously constant. They take the differential change of portfolio value Π to be

$d\Pi = -dc + \Delta_t\, dS_t$,

which misses the effect arising from the differential change in Δ_t. The ability to construct a perfectly hedged portfolio relies on the assumption of continuous trading and continuous asset price path. It has been commonly agreed that the assumed Geometric Brownian process of the asset price may not truly reflect the actual behavior of the asset price process. The asset price may exhibit jumps upon the arrival of a sudden news in the financial market. The interest rate is widely recognized to be fluctuating over time in an irregular manner rather than being constant. For an option on a risky asset, the interest rate appears only in the discount factor so that the assumption of constant/deterministic interest rate is quite acceptable for a short-lived option. The Black–Scholes pricing approach assumes continuous hedging at all times. In the real world of trading with transaction costs, this would lead to infinite transaction costs in the hedging procedure.

taken from here

← PREVIOUS NEXT →